Bayesian Model Selection in Forecasting International Migration: Simple Time Series Models and Their Extensions

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Plan of the presentation

1. Bayesian forecasting: general remarks
2. Bayesian model selection: introduction
   - Simple stochastic processes – ARMA(1,1) sub-models
   - Models with changes in conditional variance
   - Forecasting by analogy to other migration flows
4. Robustness of forecasts against selected changes in the prior distributions
5. Concluding remarks
1. Bayesian forecasting: general remarks

Predictive distribution of forecasted values $x^F$ given data $x$, weighted by posterior distributions of parameters $\theta$, equals:

$$p(x^F|x) = \int_{\Theta} p(x^F|x, \theta) \cdot p(\theta|x) \, d\theta,$$

where, by the Bayes' Theorem (1763):

$$p(\theta|x) = p(x|\theta) \cdot p(\theta)/p(x); \quad p(x) = \int_{\Theta} p(x|\theta) \cdot p(\theta) \, d\theta$$

Posterior $p(\theta|x)$ is proportional to data likelihood $p(x|\theta)$ and prior distribution $p(\theta)$, the latter denoting subjective expert knowledge (judgement) on the model parameters.

Probability in the Bayesian statistics is also a subjective concept, interpreted as a measure of belief.
2. Bayesian model selection: introduction

Theory: Uncertainty of models

- Let $M_1, \ldots, M_m$ be mutually exclusive (not nested) models adding up to a finite space of possible models, $\mathbf{M}$.

- Let $p(M_1), \ldots, p(M_m)$ be the models’ prior probabilities, e.g.:
  - Flat prior (equal probabilities): $p(M_1) = \ldots = p(M_m)$
  - “Occam’s razor” prior, favouring simpler models with smaller numbers of parameters, $l_i$: $p(M_i) \propto 2^{-l_i}$

- For forecasting, a model with highest posterior probability is selected on the basis of the Bayes’ Theorem:

$$p(M_i|x) = \frac{p(M_i) \cdot p(x|M_i)}{\sum_{k \in \mathbf{M}} \{p(M_k) \cdot p(x|M_k)\}}$$

[Hoeting et al., 1999; Osiewalski, 2001]
2. Bayesian model selection: introduction

Practice: Numerical computations

- Model Choice via Markov Chain Monte Carlo [Carlin and Chib, 1995] implemented in WinBUGS 1.4 software [Spiegelhalter et al., 2003]
- Method: iterative sampling from full conditional distributions for model-specific parameters $\theta_i$ and the model index $\mu$:

$$
p(\theta_i | \theta_{j \neq i}, \mu, x) \propto \begin{cases} 
p(x | \theta_i, \mu = i) \cdot p(\theta_i | \mu = i) & \text{for } \mu = i \\
p(\theta_i | \mu \neq i) & \text{for } \mu \neq i 
\end{cases}
$$

$$
p(\mu = i | \theta, x) = \frac{p(x | \theta_i, \mu = i) \cdot p(M_i) \cdot \prod_{j \in M} p(\theta_j | \mu = i)}{\sum_{k \in M} \left[ p(x | \theta_k, \mu = k) \cdot p(M_k) \cdot \prod_{j \in M} p(\theta_j | \mu = k) \right]^+}
$$

- Parameters $\theta_i$ are sampled either from full conditionals if $\mu = i$, or from linking densities (“pseudo-priors”) otherwise
- Iterations before reaching convergence are discarded
2. Bayesian model selection: introduction

Rationale for migration forecasting applications

- Features of the Bayesian approach:
  - The stochastic character ensures the formality of inference, with key focus on the uncertainty issue
  - The a priori expert judgement is allowed, which can supplement small-sample information (important for many time series of European migration) [cf. Willekens, 1994]

- Formal model selection techniques:
  - One way of assessing the uncertainty of model specification
  - When used with appropriate priors (e.g., “Occam’s razor”), can answer the question on simplicity versus complexity in the population forecasting models [cf. Ahlburg, 1995; Smith, 1997]
3. Forecasts of Polish–German migration

Aim
To forecast long-term migration between Germany and Poland for 2005–2015 in different modelling frameworks

Data
- Forecasted variable – logarithms of emigration rates per 1,000 population of the sending country:
  \[ m_t = \ln\left(\frac{Migt}{Pop_t} \times 1,000\right) \]
- Data sources: population stocks – Eurostat
  migration – Destatis (German data)
- Polish population stocks include post-census adjustment
3. Forecasts of Polish–German migration

a) Simple stochastic processes – ARMA(1,1) sub-models

- $M_1: m_t = c + \varepsilon_t$ [oscillations around a constant]
- $M_2: m_t = c + m_{t-1} + \varepsilon_t$ [random walk with drift]
- $M_3: m_t = c + \phi m_{t-1} + \varepsilon_t$; $\phi \not\in \{0, 1\}$ [AR(1) process]
- $M_4: m_t = c - \theta \varepsilon_{t-1} + \varepsilon_t$; $\theta \neq 0$ [MA(1) process]
- $M_5: m_t = c + \phi m_{t-1} - \theta \varepsilon_{t-1} + \varepsilon_t$; $\phi, \theta \neq 0$ [ARMA(1,1)]

Random term: $\varepsilon_t \sim \text{iid } N(0, \sigma^2)$
Sample: 1991–2004 ($N=14$)

Priors: constants $c \sim N(0, 100^2)$ diffuse (hardly informative)
$\phi, \theta \sim N(0.5, 1^2)$: processes likely stationary / time-reversible
Low precision: $\tau_{PL\rightarrow DE} = \sigma^{-2} \sim \Gamma(0.25, 0.25)$; $\tau_{DE\rightarrow PL} \sim \Gamma(4, 0.4)$
3. Forecasts of Polish–German migration

b) AR(1) extensions: non-constant conditional variance

General model: $m_t = c + \phi m_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma_t^2)$

- $M_5$: $\sigma_t^2 = \sigma^2$ [reference model, constant variance]
- $M_6$: $\sigma_t^2 = k + \alpha \cdot \varepsilon_{t-1}^2$ [AR(1)–ARCH(1) process]
- $M_7$: $\sigma_t^2 = k + \alpha \cdot \varepsilon_{t-1}^2 + \beta \cdot \sigma_{t-1}^2$ [AR(1)–GARCH(1,1)]
- $M_8$: $\ln(\sigma_t^2) = k + \gamma \cdot \ln(\sigma_{t-1}^2) + \zeta_t$ [simple stochastic volatility, SV]

Deterministic ($M_6$–$M_7$) vs. stochastic ($M_8$) changes in variance

2nd random term: $\zeta_t \sim \text{iiN}(0, \rho^2)$  
Sample: 1985–2004 ($N=20$)

Priors: $c, \phi$ as before, other concentrated for computational reasons: $\alpha, \beta, \gamma \sim \Gamma(10, 20)$; $k \sim \Gamma(1, 0.1)$; $1/\rho^2 \sim \Gamma(10, 1)$
3. Forecasts of Polish–German migration
c) Models with analogy to Iberian migration flows

Idea: to capture institutional changes, e.g., post-accession opening of the Western EU labour markets [Kupiszewski, 1998]

- $M_{10}: m_t = c + \varepsilon_t$ [reference model, no analogy]
- $M_{11}: m_t = c + a \cdot m_{PT}^{t-18} + b \cdot 1_{t=2002} + \varepsilon_t$ [Portugal]
- $M_{12}: m_t = c + a \cdot m_{ES}^{t-18} + \varepsilon_t$ [Spain]
- $M_{13}: m_t = c + a \cdot m_{IB}^{t-18} + b \cdot 1_{t=2002} + \varepsilon_t$ [both countries]

Rationale: timing of EU accession, system transformation

Random term: $\varepsilon_t \sim AR(1)$

Sample: 1992–2004 ($N=13$)

Priors: $c, \phi, \tau$ as before, $a \sim N(0.5, 1^2)$ – a positive analogy
### 3. Forecasts of Polish–German migration

#### Bayesian model selection for three proposed classes M

Model posteriors $p(M_i | x)$ under “Occam’s razor” priors, $p(M_i) \propto 2^{-li}$

<table>
<thead>
<tr>
<th>Migration flow</th>
<th>Subclasses of ARMA(1,1)</th>
<th>Extensions of variance</th>
<th>Models with analogy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_1$ $M_2$ $M_3$ $M_4$ $M_5$</td>
<td>$M_6$ $M_7$ $M_8$ $M_9$</td>
<td>$M_{10}$ $M_{11}$ $M_{12}$ $M_{13}$</td>
</tr>
<tr>
<td>Poland → Germany</td>
<td>0.42 0.29 0.14 0.11 0.03</td>
<td>0.12 0.10 0.05 0.74</td>
<td>0.69 0.07 0.14 0.10</td>
</tr>
<tr>
<td>Germany → Poland</td>
<td>0.23 0.49 0.16 0.08 0.03</td>
<td>0.71 0.02 0.00 0.27</td>
<td>0.88 0.01 0.09 0.02</td>
</tr>
<tr>
<td>Model priors $p(M_i)$</td>
<td>0.31 0.31 0.15 0.15 0.08</td>
<td>0.50 0.25 0.13 0.13</td>
<td>0.40 0.20 0.20 0.20</td>
</tr>
</tbody>
</table>

- Simple random models (oscillations / random walks)
- Either constant or stochastic conditional variance
- No linear analogies supported by the data
3. Forecasts of Polish–German migration

Overview of selected empirical results

Migration from Poland to Germany: selected distributions, forecast for 2005

Migration from Germany to Poland: selected distributions, forecast for 2005

Grey lines – distributions \textit{a priori}, black – \textit{a posteriori}.

Highest \textit{ex post} errors for $M_1$, lowest – for $M_2$
### 3. Forecasts of Polish–German migration

**Exp($m_t$) forecasts from the selected models, 2005–2015**

<table>
<thead>
<tr>
<th>Model</th>
<th>Forecasted exp($m_{2005}$)</th>
<th>Forecasted exp($m_{2010}$)</th>
<th>Forecasted exp($m_{2015}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10% Median 90%</td>
<td>10% Median 90%</td>
<td>10% Median 90%</td>
</tr>
<tr>
<td><strong>Poland → Germany</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_1$: oscillation</td>
<td>1.77 2.57 3.72</td>
<td>1.78 2.57 3.71</td>
<td>1.78 2.57 3.72</td>
</tr>
<tr>
<td>$M_6$: AR(1)–SV</td>
<td>2.64 3.42 4.41</td>
<td>1.74 3.31 7.18</td>
<td>1.60 3.28 8.40</td>
</tr>
<tr>
<td>$M_{10}$: no analogy</td>
<td>1.96 2.99 4.41</td>
<td>1.71 2.62 4.04</td>
<td>1.71 2.63 4.05</td>
</tr>
<tr>
<td><strong>Germany → Poland</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_2$: RWD</td>
<td>0.91 1.25 1.71</td>
<td>0.48 1.18 2.90</td>
<td>0.28 1.12 4.36</td>
</tr>
<tr>
<td>$M_6$: AR(1)</td>
<td>0.90 1.26 1.76</td>
<td>0.69 1.24 2.36</td>
<td>0.63 1.24 2.71</td>
</tr>
<tr>
<td>$M_6$: AR(1)–SV</td>
<td>0.94 1.25 1.67</td>
<td>0.72 1.21 2.24</td>
<td>0.68 1.21 2.62</td>
</tr>
<tr>
<td>$M_{10}$: no analogy</td>
<td>0.80 1.12 1.55</td>
<td>0.69 1.01 1.48</td>
<td>0.69 1.01 1.48</td>
</tr>
</tbody>
</table>

- Median trajectories plausible, indicate stabilisation
- Limits of 80-percent predictive intervals reasonable, except for the (likely) non-stationary models (RWD/AR)
4. Robustness against changes in priors

a) Uniform model priors $p(M_i)$ instead of the “Occam’s razor”
   
   – Results for ARMA(1,1) sub-models: the same ones selected, although with different posterior probabilities $p(M_i|\mathbf{x})$

b) Alternative prior distributions for $\theta$, used as a reference: the non-informative ones [Jeffreys, 1961]
   
   – In practical applications, “hardly informative” priors can be used for computational convenience: [Congdon, 2003]

   For structural parameters: $N(0, D^2)$, where $D$ is a big number
   
   For precision $\tau = \sigma^{-2}$: $\Gamma(a, a)$, with small parameters $a$

   – Under $D = 100$ and $a = 0.001$, the obtained forecasts for oscillations, random walks and AR(1) differ from the “informative” ones with respect to uncertainty estimates
4. Robustness against changes in priors

Results: informative vs. hardly informative priors on precision

Without a priori assumptions on low precision $\tau$, in many cases the 80-percent intervals are narrower than within-sample variability.
5. Concluding remarks

• Bayesian model selection techniques allow for identifying models with the highest data support and for assessing uncertainty on various levels, including model specification.

• Empirical results: simple, unstructured models preferred.

• Selection of oscillations, random walks and stochastic volatility models confirms a hardly predictable character of both migration rates and their uncertainty measures.

• Given the shortness of data series, the results are not robust against changes in priors (especially for precision).

• However, without assuming low precision \textit{a priori}, the predictive intervals would be in many cases too narrow as for such an uncertain phenomenon as migration.
Thank you for your attention!

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